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GRILLAGE BEAMS ON ELASTIC FOUNDATION

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GRILLAGE BEAMS ON ELASTIC FOUNDATION[1]

Bernard Drapkin[a]

SYNOPSIS

The geometric and elastic quantities influencing the maximum moment under a concentrated load for a beam on an elastic, homogeneous, isotropic halfspace, are represented by parametric curves. A numerical determination of the spring constant, k, is effected for any particular case by equating the maximum moment determined from the curves to that of a beam resting on a spring foundation. The spring theory is subsequently applied using this readily available k; this would otherwise require experimental determination each time any physical quantity is varied.

It is found that for the usual beams employed in practice there is a slight reduction in the maximum moment from that of the infinitely long beam. For foundations of high Young's modulus it is quantitatively determined that the maximum moment is less than for foundations of low Young's modulus. The effect of increasing the flexural rigidity of the beam is to increase the maximum moment.

A simple formula is presented for steel beams on rock or average concrete loaded by a single concentration at the center. If the ratio of Young's moduli of beam and foundation is 15, the maximum moment $= 2/3$ Load $\sqrt[3]{\dfrac{\text{Mom. of Inertia}}{\text{1/2 flange width}}}$. The coefficient 2/3 is dimensionless.

The maximum moment for elastic-rock foundations determined by the procedure of this paper, is of the order of one-half that obtained when the usual practical design assumption of uniform pressure is employed. The maximum foundation pressure is of the order of twice the uniform foundation pressure. The assumption of uniform pressure therefore gives an understressed beam and an overstressed foundation under the concentration.

INTRODUCTION

This paper presents a solution of the grillage beam problem which is compatible with usual assumptions of the theory of elasticity and theoretical soil mechanics.

The simplifying assumptions used in actual current and past practice to analyze and design these important foundation members are:

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1. This paper is based on a dissertation submitted to the Polytechnic Institute of Brooklyn in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the field of applied mechanics.

a) The concentrated column load imposed on the beam is assumed to result in uniform pressure distribution on the supporting medium regardless of the relative stiffness of the foundation and beam, and independent of geometrical quantities such as beam length, beam width, foundation width and foundation depth;

b) The selection of the beam is made by keeping the pressure below a certain specified value and computing the maximum moment in the beam based on assumption (a) and the elementary beam theory.

Most of the existing literature dealing with the subject of beams on elastic foundation assumes a known predetermined modulus of foundation k, the use of which simplifies the solution of the linear fourth order differential equation: $E_b I \frac{d^4y}{dx^4} + ky = p(x)$.* The basic problem is the infinite beam on a spring foundation of modulus k, subjected to a single concentrated load. From this solution, by means of the superposition principle, solutions for various loading conditions have been determined. The direct solution of the finite beam on a spring foundation of modulus k is unwieldy due to the boundary conditions; the superposition procedure used extensively by Hetenyi[2] is more elegant.

A Fourier integral solution of the infinitely long beam carrying a concentrated load and resting on an elastic foundation has been derived by Biot.[3] Biot obtained values for the foundation modulus, k, of the spring theory by equating the maximum moment in the infinite beam resting on the elastic foundation to the maximum moment in the same infinite beam on a spring foundation obtained from the elementary theory. Using this equivalent value of k, one can determine the moment, shear, deflection and pressure distribution along the beam using the elementary spring theory.

Problem

The basic structural element of a foundation grillage is a steel beam in full contact with the supporting medium and transmitting a heavy concentrated load. Investigation of beams of finite length resting on an elastic foundation will be made to determine the maximum moment in the beam. From this result the equivalent spring constant k is determined. The effects of the length, width, flexural rigidity of the beam and the modulus of elasticity and Poisson's ratio of the foundation on the equivalent spring constant for beams of any length, are presented in this paper by means of parametric curves. Comparison will be made with the results obtained by means of the usual design assumption of uniform foundation pressure under grillage beams.

Assumptions of the Theory of Elasticity used in Paper

a) The supporting medium is an elastic foundation, homogeneous, isotropic, semi-infinite in depth and extent, without body forces, and characterized by the elastic constants, Young's modulus E and Poisson's ratio ν.

b) The elementary beam theory is applicable. The beam has flexural rigidity E_bI and negligible flexibility across its width of flange 2b.

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* See Notation

2. "Beams on Elastic Foundation," by M. Hetenyi, U. of Michigan Press, Ann Arbor, Michigan, 1946, p. 9.
3. "Bending of an Infinite Beam on an Elastic Foundation," by M. A. Biot, J. Appl. Mech., Vol. 4, 1937, pp. A1-A7.

c) The elastic foundation is always in contact with the beam and can support tension as well as compression. Young's modulus is assumed to be the same constant in tension and compression.

d) There are negligible shearing stresses at the contact surface.

The first assumption in (c) is unnecessary when the contact pressure is sufficiently large to keep the beam in contact with the foundation. That is, the beam under its own weight induces a certain amount of compression over the entire contact surface; this initial compression may be everywhere greater than any induced negative pressure due to the concentrated load.

The use of the elementary beam theory, within its limitations, is valid for grillage beams as for beams supported in other ways. The depth of the beams considered must be small compared with the length, since the elementary theory assumes the shearing strains to have negligible effect on the axial strains. Straight prismatic beams are assumed to have proportions that preclude failure by twisting, lateral collapse and local wrinkling. The deflection, slope, moment and shear are proportional to the concentrated load since the basic differential equation is linear. The superposition principle and reciprocal theorem are therefore applicable.

Solution

Biot[4] showed that the moment at any point in an infinitely long beam carrying a concentrated load P and resting on an elastic foundation is:

$$M(x) = \frac{Pc}{\pi} \int_0^\infty \frac{\alpha \cos\left(\alpha \frac{x}{c}\right) d\alpha}{\alpha^3 + \psi(\beta)} \qquad (1)*$$

The superposition principle and Maxwell's reciprocal theorem are used to determine the free end effects for beams of finite length. Unknown moments M_o and unknown shears Q_o are applied to those points on the infinite beam corresponding to the end points of the finite beam. The magnitudes and directions of M_o and Q_o are then determined so that the moments and shears in the infinite beam become zero at those points. The moment at any point in the finite beam is then expressed by adding the moment due to the concentrated load, the moment due to the moment M_o and the moment due to the shear Q_o.

Differentiating under the integral sign with respect to x in (1), the shear at any point in the infinite beam due to a concentrated load at the origin is obtained:

$$V(x) = \frac{dM}{dx} = P\,c\,\frac{d\Phi(x)}{dx} \qquad (2)$$

4. "Bending of an Infinite Beam on an Elastic Foundation," by M. A. Biot, J. Appl. Mech., Vol. 4, p. 1A, 1937.

* See Notation

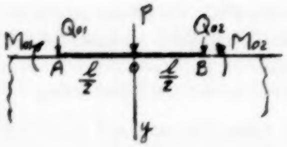
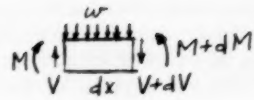


Fig. 1



Beam Element

The beam sign convention used is: Shear taken positive when up on the left end of OB in Fig. 1.

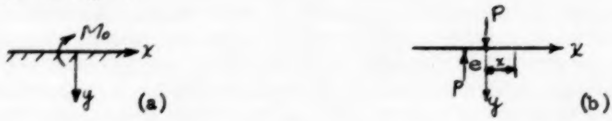


Fig. 2

Fig. 2(a) is a limiting case of Fig. 2(b).
$M_o = Pe =$ constant couple, as $e \to 0$, $Pe \to M_o$.

$$\underset{(\text{due } M_o \text{ at } 0)}{M(x)} = Pc\,\Phi(x) - Pc\,\Phi(x+e) = Pc\left[\Phi(x) - \Phi(x+e)\right]$$

$$M(x) = M_o c\left[\frac{\Phi(x) - \Phi(x+e)}{e}\right], \quad \text{and in the limit as } e \to 0,$$

$$\underset{(\text{due } M_o \text{ at } 0)}{M(x)} = -M_o\, c\, \frac{d\Phi(x)}{dx} \tag{3}$$

$$\underset{(\text{due } M_o \text{ at } 0)}{V(x)} = \frac{dM(x)}{dx} = -M_o\, c\, \frac{d^2\Phi(x)}{dx^2} \tag{4}$$

M' and V' are the moment and shear at A due to P.

$$M' = Pc\,\Phi(\ell/2) \tag{5}$$

$$V' = -Pc\,\frac{d\Phi(\ell/2)}{dx} \tag{6}$$

M'' and V'' are the moment and shear at A due to M_{o1}, M_{o2}, Q_{o1}, Q_{o2}.

$$M'' = Q_o\, c\left[\Phi(0) + \Phi(\ell)\right] - M_o\, c\left[\frac{d\Phi(0)}{dx} + \frac{d\Phi(\ell)}{dx}\right] \tag{7}$$

$$V'' = Q_o c\left[\frac{d\Phi(0)}{dx} - \frac{d\Phi(\ell)}{dx}\right] - M_o\, c\left[\frac{d^2\Phi(0)}{dx^2} - \frac{d^2\Phi(\ell)}{dx^2}\right] \tag{8}$$

Now, $M' + M'' = 0$ and $V' + V'' = 0$, since the moment and shear are zero at A.

$$Pc\,\Phi(\ell/2) + Q_0 c\left[\Phi(0) + \Phi(\ell)\right] - M_0 c\,\frac{d\Phi(\ell)}{dx} + M_0/2 = 0 \qquad (9)$$

$$Pc\,\frac{d\Phi(\ell/2)}{dx} + Q_0 c\,\frac{d\Phi(\ell)}{dx} - Q_0/2 + M_0 c\left[\frac{d^2\Phi(0)}{dx^2} - \frac{d^2\Phi(\ell)}{dx^2}\right] = 0 \qquad (10)$$

Now, using $\zeta = x/c$ as dimensionless coordinate,

$$\frac{d\zeta}{dx} = \frac{1}{c}\,, \quad \frac{d\Phi}{dx} = \frac{d\Phi}{d\zeta}\cdot\frac{d\zeta}{dx} = \frac{1}{c}\,\frac{d\Phi}{d\zeta}\,, \quad \frac{d^2\Phi}{dx^2} = \frac{1}{c^2}\,\frac{d^2\Phi}{d\zeta^2}\,.$$

Hence (9) and (10) become:

$$P\Phi(\ell/2c) + Q_0\left[\Phi(0) + \Phi(\ell/c)\right] - \frac{M_0}{c}\left[\Phi'(\ell/c) - 1/2\right] = 0 \qquad (9a)$$

$$P\Phi'(\ell/2c) + Q_0\left[\Phi'(\ell/c) - 1/2\right] + \frac{M_0}{c}\left[\Phi''(0) - \Phi''(\ell/c)\right] = 0 \qquad (10a)$$

This pair of simultaneous equations can be solved for Q_0 and M_0:

$$Q_0 = P\,\frac{-\Phi'(\ell/2c)\left[\Phi'(\ell/c) - \frac{1}{2}\right] - \Phi(\ell/2c)\left[\Phi''(0) - \Phi''(\ell/c)\right]}{\left[\Phi(0) + \Phi(\ell/c)\right]\left[\Phi''(0) - \Phi''(\ell/c)\right] + \left[\Phi'(\ell/c) - \frac{1}{2}\right]^2} = P\bar{\lambda} \qquad (11)$$

$$M_0 = Pc\,\frac{\Phi(\ell/2c) + \bar{\lambda}\left[\Phi(0) + \Phi(\ell/c)\right]}{\Phi'(\ell/c) - \frac{1}{2}} \qquad (12)$$

Here the prime ' denotes differentiation with respect to ζ,

i. e., $' \equiv d/d\zeta$

An alternate form for M_0 which is more suitable numerically when dealing with higher values of the argument $\ell/2c$ is found by eliminating Q_0 between (9a) and (10a), giving:

$$M_0 = Pc\,\frac{\Phi(\ell/2c)\left[\Phi'(\ell/c) - \frac{1}{2}\right] - \Phi'(\ell/2c)\left[\Phi(0) + \Phi(\ell/c)\right]}{\left[\Phi''(0) - \Phi''(\ell/c)\right]\left[\Phi(0) + \Phi(\ell/c)\right] + \left[\Phi'(\ell/c) - \frac{1}{2}\right]^2} \qquad (12a)$$

$$\underset{\substack{\text{finite}\\ \text{beam}}}{M_\zeta} = Pc\,\Phi(\zeta) + Q_0\,c\left[\Phi(\ell/2c - \zeta) + \Phi(\ell/2c + \zeta)\right] - M_0\left[\Phi'(\ell/2c - \zeta) + \Phi'(\ell/2c + \zeta)\right] \qquad (13)$$

The first term is the moment in an infinitely long beam and the remaining terms represent the effect of the shears and moments which have been determined by (11) and (12). The sum of these terms gives the moment at any station in the finite beam. It is necessary to evaluate the following types of integrals:

$$\Phi(\zeta)=\frac{1}{\pi}\int_0^\infty \frac{\alpha\cos\alpha\zeta\, d\alpha}{\alpha^3+\psi(\beta)} \tag{14}$$

$$\Phi'(\zeta)= -\frac{1}{\pi}\int_0^\infty \frac{\alpha^2 \sin\alpha\zeta\, d\alpha}{\alpha^3+\psi(\beta)} \tag{15}$$

$$\Phi''(\zeta)= -\frac{1}{\pi}\int_0^\infty \frac{\alpha^3 \cos\alpha\zeta\, d\alpha}{\alpha^3+\psi(\beta)} \tag{16}$$

Integrals (14) and (15) are convergent whereas (16) is a divergent integral.

From physical considerations $\Phi(\zeta)$ represents the dimensionless quantity by which Pc is multiplied to obtain the moment at any point in an infinite beam due to a concentrated load at the origin.

$\Phi'(\zeta)$ represents the dimensionless quantity by which P is multiplied to obtain the shear at any point in an infinite beam due to a concentrated load, P, at the origin. $\Phi'(\zeta)$ also represents the dimensionless quantity by which M_o is multiplied to obtain the moment at any point in an infinite beam due to M_o at the origin.

$\Phi''(\zeta)$ represents the dimensionless quantity by which M_o/c is multiplied to obtain the shear at any point in an infinite beam due to M_o at the origin.

In all numerical work the boundary conditions that the moment and shear are zero at the free ends of the beam, $\zeta = \ell/2c$, are applied to give an additional check on the work.

The maximum moment, M, occurs under the load at $\zeta = 0$ and is given by the expression:

$$M_{max} = Pc\Phi(0) + 2Q_o\, c\Phi(\ell/2c) - 2M_o\Phi'(\ell/2c) \tag{17}$$

$\psi(\beta)$ is a relation between the maximum load per unit length on the foundation and the average deflection across the loaded width. Inasmuch as $\psi(\beta)$ tends to unity as α tends to infinity, the $\Phi''(\zeta)$ integral (16) is divergent. However, oscillating divergent infinite integrals can be considered as oscillating divergent series and can sometimes be summed, whereas non-oscillating divergent series cannot be summed.

Add and subtract the expression $\psi(\beta)$ in the numerator of (16):

$$\int_0^\infty \frac{\alpha^3\cos\alpha\zeta\, d\alpha}{\alpha^3+\psi(\beta)} = \int_0^\infty \frac{\left[\alpha^3+\psi(\beta)-\psi(\beta)\right]\cos\alpha\zeta\, d\alpha}{\alpha^3+\psi(\beta)}$$

$$= \underbrace{\int_0^\infty \cos\alpha\zeta\, d\alpha}_{\text{I}} - \underbrace{\int_0^\infty \frac{\psi(\beta)\cos\alpha\zeta\, d\alpha}{\alpha^3+\psi(\beta)}}_{\text{II}} \tag{16a}$$

Treat Integral I using Cesaro's method as indicated by Hardy[5] and others:

5. "Divergent Series," by G. H. Hardy, Oxford at the Clarendon Press, 1949, p. 11.

$$\text{Given} \quad \int_a^{\infty} f(\alpha)\, d\alpha \qquad \text{divergent} \tag{16b}$$

$$\text{form} \quad \lim_{T\to\infty} \int_a^{T} (1 - \alpha/T)\, f(\alpha)\, d\alpha = S \tag{16c}$$

$$\text{then} \quad S = \int_a^{\infty} f(\alpha)\, d\alpha \,. \tag{16d}$$

Now $f(\alpha) = \cos \alpha\, \zeta$ and the lower limit $a = 0$.
It is further stipulated that ζ is greater than some finite quantity which does not tend to zero.

$$\text{Evaluate} \int_0^{T} (1 - \alpha/T) \cos\alpha\zeta\, d\alpha \quad \text{by parts:}$$

$$\int_0^{T} \alpha \cos\alpha\zeta\, d\alpha = \frac{1}{\zeta}\left(T \sin\zeta T - 0\sin\zeta.0 + \frac{1}{\zeta}\cos\zeta T - \frac{1}{\zeta}\cos\zeta.0\right)$$

$$\int_0^{T} \cos\alpha\zeta\, d\alpha = \frac{1}{\zeta}(\sin\zeta T - 0)\,.$$

$$\therefore \left(\frac{1}{\zeta}\sin\zeta T - \frac{1}{\zeta}\sin\zeta T - \frac{1}{T}\frac{1}{\zeta^2}\cos\zeta T + \frac{1}{\zeta T}\right) \to 0, \text{ as } T\to\infty,$$

Since $\cos \zeta.0 = 1$, $\sin\zeta.0 = 0$; and $\sin\zeta\ T$ and $\cos\zeta\ T$ are bounded for all T.

$\therefore$ the divergent integral $S = \int_0^{\infty} \cos\alpha\zeta\, d\alpha \sim 0$.

Note that the sumbol $\sim$ means 'corresponds' since the sign of equality implies convergence.

Since integral I corresponds to zero, the divergent integral on the left hand side of (16a) is replaced by the convergent integral II.

The same results can be shown by Abel's method of evaluating divergent integrals; or, by using the exponential form indicated by Hardy.[6] Furthermore, integral I in equation (16a) can also be shown to correspond to zero by introducing the factor $e^{-a\alpha}$ on both sides of (16a) and allowing "a" to tend to zero.

The complicated function $\psi(\beta)$ was derived by Biot as an asymptotic expression for $\beta < 0.1$, and tabular values were stated for $\beta > 0.1$. In the present paper, to facilitate the required evaluation of the convergent integrals (14), (15), and (18) for the finite beam, $\psi(\beta)$ is now defined as the hyperbolic curve $1 + \frac{a}{\beta}$. The value of a is determined by using the method of least squares.[7] See Fig. 3. It can be shown that a small variation in the curve fit constant a has a negligible effect on the values of integrals of the types required. A 6% variation in choosing $a = 0.3$ instead of 0.33, for example, results in only a 1/2 of 1% variation in the values of integrals of the types (14), (15) and (18).

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6. "Divergent Series," by G. H. Hardy, Oxford at the Clarendon Press, 1949, p. 12.
7. "Graphical and Mechanical Computation," by J. Lipka, John Wiley and Sons, New York, 1918, p. 126.

FITTING HYPERBOLIC EXPRESSION FOR $\psi(\beta)$ TO BIOT'S TABULAR VALUES FOR HIGH β AND THE CORRECTED LOGARITHMIC EXPRESSION FOR LOW β.

$$\psi(\beta)_{log.} = \frac{\pi}{2\beta}\left(\log_e \frac{1}{\beta} + 0.923\right)^{-1}$$

$$\psi(\beta)_{hyp} = 1 + \frac{a}{\beta}$$

	β	$\psi(\beta)_{log.}$	$\frac{1}{\beta}$	$\frac{1}{\beta}\psi(\beta)_{log.}$	$\left(\frac{1}{\beta}\right)^2$	$\psi(\beta)_{hyp.}$
	0.10	4.87	10.0	48.7	100	4.3
	0.09	5.23	11.1	58.1	123	4.7
	0.08	5.68	12.5	71.0	156	5.1
	0.07	6.28	14.3	89.8	204	5.7
	0.06	7.01	16.7	117	279	6.5
	0.05	8.01	20.0	160	400	7.6
	0.04	9.54	25.0	239	625	9.3
	0.03	11.8	33.3	393	1110	11.0
	0.02	16.3	50.0	815	2500	17.0
	$\Sigma=$	74.7	192.9	1992	5497	
	0.01	29				34
Biot's Table	0.5	1.9				1.7
	1	1.42				1.3
	3	1.13				1.1
	8	1.04				1.04
	∞	1				1

$\psi(\beta)$ is asymptotic to the vertical axis and the horizontal line $\psi(\beta) = 1$.

Fit the hyperbolic curve in the lower ranges of β to the logarithmic expression and the correspondence is good in the upper ranges of β.

Method of Least Squares

$$\frac{\partial}{\partial a}\Sigma\left(\psi(\beta)_{log.} - 1 - \frac{a}{\beta}\right)^2 = 0$$

$$\therefore \Sigma\left(\psi(\beta)\cdot\frac{1}{\beta}\right) = 1\cdot\Sigma\frac{1}{\beta} + a\Sigma\left(\frac{1}{\beta}\right)^2$$

$1992 = 192.9 + 5497\ a$

$a = 0.327$ say 0.33

Method of Averages

$74.7 = 9 + 192.9\ a$

$a = 0.34$

$$\Sigma\psi(\beta) = n\times 1 + a\Sigma\frac{1}{\beta}$$

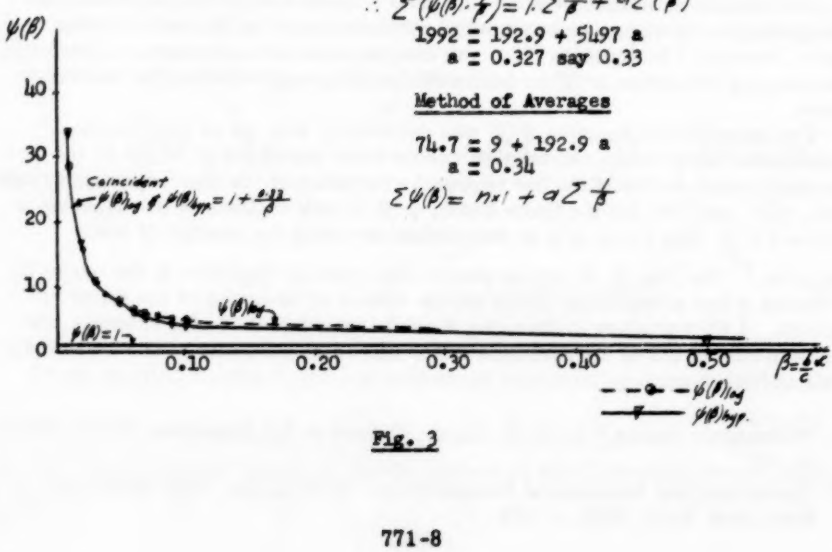

Fig. 3

This simple expression for $\psi(\beta)$ permits closed form evaluations of integrals (14) and (18) when the trigonometric term vanishes. The polynomial expressions obtained in the denominators of the integrands in (14), (15) and (18), moreover, makes possible the application of the method of residues.

The convergent integrals (14), (15) and (18) are difficult to evaluate directly when $\mathcal{J} = \ell/c$, $\ell/2c$ is of the order of 10, due to the rapid oscillations of the integrands. The computational work can be considered reduced by the application of Euler's transformation of series, which greatly accelerates the slow convergence of the required integrals.

A more effective means of evaluation is available through the use of the method of residues. The values of the physical constants which are selected are based on representative designs of grillage beams.[8,9] See Table 1 below.

TABLE 1

PROPERTIES OF STANDARD COLUMN GRILLAGES

BASED ON KETCHUM'S TABLES [8]

$C = 1.06 \qquad \nu = \frac{1}{4}$

Beam	Length	I (in^4)	b (in)	c (in)	b/c		$\ell/2c$	
8 I 20.5	3'-0"	60.2	2.04	7.62	0.267		2.36	
12 I 45	4'-3"	284.1	2.68	11.6	0.232		2.20	
15 I 55	5'-5"	508.7	2.87	13.8	0.208		2.36	
15 I 65	5'-10"	632.1	3.04	14.6	0.209	0.225	2.40	2.35
20 I 90	7'-5"	1550.3	3.56	18.7	0.191		2.38	
24 I 120	8'-11"	3010.8	4.02	22.4	0.180		2.40	

$E = 2 \times 10^6$ #/in^2 for average reinf. concrete, brickwork, sand stone

8 I 20.5	3'-0"	60.2	2.04	36.8	0.0552		0.490	
12 I 45	4'-3"	284.1	2.68	56.0	0.0480		0.456	
15 I 55	5'-5"	508.7	2.87	66.5	0.0431	0.044	0.490	0.49
15 I 65	5'-10"	632.1	3.04	70.2	0.0431		0.497	
20 I 90	7'-5"	1550.3	3.56	90.0	0.0395		0.494	
24 I 120	8'-11"	3010.8	4.02	107.0	0.0374		0.497	

$E = 0.018 \times 10^6$ #/in^2 based on sand curves[11]

For the standard column grillages in Ketchum's tables the $\ell/2c$ values appear to be fairly constant. For charts, however, the beam length is varied in order to illustrate the free end effects of the finite beam. As ℓ is increased, the end effects become negligible and the beam behaves as if infinitely long.

c = functional relation or fundamental length $= \left[c(1-\nu^2)\dfrac{E_b I}{E\, b} \right]^{1/3}$

$2b$ = beam width

8. "Structural Engineers' Handbook," by M. S. Ketchum, McGraw-Hill Book Company, New York, 1924, Part II, pp. 275-283.
9. "Progress Report of Special Committee to Codify Present Practice on Bearing Values of Soils for Foundations," A.S.C.E. Papers and Discussions, Vol. 43, 1917, p. 1176.

Numerical values of b/c which correspond to the particular values of Young's modulus E for the foundation, are introduced into the expression for $\psi(\beta)$. Fourth degree polynomials in α are then obtained in the denominators of the integrands.

For $b/c = 0.205$, which corresponds to $E = 2 \times 10^6$ #/in^2, the oscillating integrals in (14), (15) and (18) are transformed into integrals containing the negative exponential $e^{-\gamma y}$ and some additional explicit trigonometric terms. Numerical evaluations of damped non-oscillating integrals are much simpler than for oscillating integrals.

The path chosen for contour integration is in the first quadrant along the real axis, quarter circle and imaginary axis. The roots of the biquadratic in the denominators are the singularities of the integrands. There is one in each quadrant and therefore only the residue of the pole in the first quadrant need be evaluated.

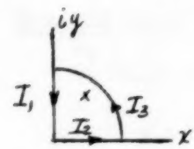

Fig. 4

pole at $a + bi$

For the integral in (14), for example:

Consider
$$\int_c \frac{z^2 e^{i\gamma z}\, dz}{z^4 + z + k_1} \tag{19}$$

where $k_1 = 1.6097561$, based on $b/c = 0.205$; i.e., $\psi(\beta) = 1 + \frac{0.33}{0.205\alpha}$.

The integral along the imaginary axis is obtained by letting $z = iy$ in (19),

$$\therefore I_1 = \int_0^\infty \frac{y^3 e^{-\gamma y}\, dy}{(y^4 + k_1)^2 + y^2} + i \int_0^\infty \frac{y^2 (y^4 + k_1)\, e^{-\gamma y}\, dy}{(y^4 + k_1)^2 + y^2}. \tag{20}$$

The integral along the real axis is obtained by letting $z = x$ in (19),

$$I_2 = \int_0^\infty \frac{x^2 \cos \gamma x\, dx}{x^4 + x + k_1} + i \int_0^\infty \frac{x^2 \sin \gamma x\, dx}{x^4 + x + k_1} \tag{21}$$

The absolute value of the integral along the quarter circle is shown to tend to zero as R tends to infinity by letting $z = R e^{i\vartheta}$ in (19). Noting that $\left|e^{mi\vartheta}\right| = 1$, and

$$\left|R^4 e^{4i\vartheta} + Re^{i\vartheta} + k_1\right| \geq \left|R^4 + R - k_1\right|$$

and $\sin\vartheta \geq \frac{2\vartheta}{\pi}$ where $0 \leq \vartheta \leq \frac{\pi}{2}$,

$$\therefore I_3 \leq \frac{iR^2}{R^4 + R - k_1} \frac{\pi}{2\gamma} \left(1 - e^{-\gamma R}\right) \quad \text{for } \gamma > 0. \tag{22}$$

The denominator is two degrees higher than the numerator in R and hence $|I_3| \to 0$ as $R \to \infty$.

$$\text{The residue} = b_1 = \frac{p(z_o)}{q'(z_o)} \qquad \text{where} \quad \frac{p(z)}{q(z)} = \frac{z^2 e^{i\Upsilon z}}{z^4 + z + k_1}$$

$$z_o = a + bi = 0.8080650 + 0.9809945\, i, \quad q'(z_o) = \left.\frac{dq}{dz}\right|_{z = z_o}$$

$$\therefore (2\pi i)\ \text{Residue} = \frac{(2\pi i)(a+bi)^2\, e^{-i\Upsilon(a+bi)}}{4(a+bi)^3 + 1} \quad . \tag{23}$$

Using the residue theorem, $\int_c f(z)\,dz = 2\pi i \Sigma K$, equate the real parts of I_1 and I_2 to the real part of $2\pi i \Sigma K$, where ΣK equals the sum of the residues at enclosed singularities. In this case, there is just one in the first quadrant.

$$\therefore \int_0^\infty \frac{\alpha^2 \cos\Upsilon\alpha\, d\alpha}{\alpha^4 + \alpha + 1.6097561} = 2\pi e^{-0.9809945\Upsilon}(0.1602625\cos 0.8080650\Upsilon - 0.1504682 \sin 0.8080650\Upsilon) - \int_0^\infty \frac{y^3\, e^{-\Upsilon y}\, dy}{(y^4 + 1.6097561)^2 + y^2} \tag{24}$$

The treatment of the integrals (15) and (18) is similar. The results are stated below:

$$-\pi \Phi'(\Upsilon) = \int_0^\infty \frac{\alpha^3 \sin\alpha\Upsilon\, d\alpha}{\alpha^4 + \alpha + 1.6097561} = 2\pi e^{-0.9809945\Upsilon}(0.2788046\cos 0.8080650\Upsilon - 0.0181060\sin 0.8080650\Upsilon) - \int_0^\infty \frac{y^4\, e^{-\Upsilon y}\, dy}{(y^4 + 1.6097561)^2 + y^2} \tag{25}$$

$$\pi \Phi''(\Upsilon) = \int_0^\infty \frac{(\alpha + 1.6097561)\cos\Upsilon\alpha\, d\alpha}{\alpha^4 + \alpha + 1.6097561}$$

$$= 2\pi e^{-0.9809945\Upsilon}(0.2881365 \cos 0.8080650\Upsilon + 0.2075301 \sin 0.8080650\Upsilon) - \int_0^\infty \frac{y^5\, e^{-\Upsilon y}\, dy}{(y^4 + 1.6097561)^2 + y^2} \tag{26}$$

For $b/c = 0.044$ and $E = 0.018 \times 10^6$ #/in^2 corresponding to the 'sand' foundation, the analagous identities follow:

$$\pi \Phi(\Upsilon) = \int_0^\infty \frac{\alpha^2 \cos\Upsilon\alpha\, d\alpha}{\alpha^4 + \alpha + 7.5} = 2\pi e^{-1.2596744\Upsilon}(0.1072136\cos 1.1719446\Upsilon - 0.1063396\sin 1.1719446\Upsilon) - \int_0^\infty \frac{y^3\, e^{-\Upsilon y}\, dy}{(y^4 + 7.5)^2 + y^2} \tag{27}$$

$$-\pi\Phi'(\zeta) = \int_0^\infty \frac{\alpha^3 \sin\zeta\alpha \, d\alpha}{\alpha^4 + \alpha + 7.5} = 2\pi e^{-1.259674\zeta}(0.2596809\cos 1.171946\zeta - 0.00830488\sin 1.171946\zeta) - \int_0^\infty \frac{y^4 e^{-\zeta y} \, dy}{(y^4 + 7.5)^2 + y^2} \qquad (28)$$

$$\pi\Phi''(\zeta) = \int_0^\infty \frac{(\alpha + 7.5)\cos\zeta\alpha \, d\alpha}{\alpha^4 + \alpha + 7.5} = 2\pi e^{-1.259674\zeta}(0.3368411\cos 1.171946\zeta + 0.2938674\sin 1.171946\zeta) - \int_0^\infty \frac{y^5 e^{-\zeta y} \, dy}{(y^4 + 7.5)^2 + y^2} \qquad (29)$$

In view of the previous discussion concerning the correspondence of $\int_0^\infty \cos\alpha\zeta \, d\alpha$ to zero, it is to be noted that the derivatives of (25) and (28) with respect to ζ yield (26) and (29). This serves as an additional check on the numerical work.

By this procedure the oscillating integrals are summed by means of numerical evaluations of damped integrals and additional explicit trigonometric terms. Thus slowly convergent oscillating integrals become rapidly convergent damped integrals. This fact arises when we consider that the larger the value of ζ the more rapidly the original functions oscillate. Rather small intervals in Simpson's rule are consequently required in order to secure a desired accuracy. For the damped integrals, however, large values of ζ cause the functions to decrease rapidly due to the negative exponential.

Table 2 gives the values of the integrals in (14), (15) and (18) for those values of ζ corresponding to ℓ/c, $\ell/2c$ and 0, which are required in order to obtain the maximum moment by the superposition procedure. This table gives the results when one considers the practical designs in Ketchum,[8] for two limiting cases of elastic foundations: one of high ($E = 2{,}000{,}000$ #/in^2) and one of low ($E = 18{,}000$ #/in^2) Young's modulus.

Parametric curves of M_{max}/Pc as ordinate against $\ell/2c$ as abscissa are shown in Fig. 5 for these two extremes of elastic foundation. The values of Young's modulus are incorporated into the parameter b/c. Hence the curves are most general and can be applied when any of the geometric and elastic quantities are varied. This is shown in the section on comparison with test results.

Infinitely long beams, or those sufficiently long so as to have negligible end effects, are represented by horizontal lines in Fig. 5. The F curves indicate that, other factors being equal, the moment in the beam is less for a foundation of large E than for one of small E. This occurs despite the fact that the coefficient F is somewhat greater for large than for small E. For $\ell/2c = 3.4$, for example, F is 0.275 as compared with 0.204. The value of c, however, is much larger for small E as it varies inversely with the cube root of E. Since this increase in c is much greater than the decrease in F, the maximum moment PcF is higher for the foundation of lower E. This is to be expected from physical considerations since the stiffer foundation should result in smaller beam deflection and consequently less moment.

This behavior is indicated by the graphs for all grillage beams. A practical example of a grillage footing on sand indicates that due to the necessity of keeping the pressure below a certain specified value, the area of the footing

TABLE 2

SAND

Closed Forms: $\Phi(0) = 0.2039169$ $\Phi''(0) = 0.6282867$

$f = \ell/2c$	$\Phi(\ell/2c)$ Eq.(14)	$\Phi(\ell/c)$ Eq.(14)	$\Phi'(\ell/2c)$ Eq.(15)	$\Phi'(\ell/c)$ Eq.(15)	$\Phi''(\ell/c)$ Eq.(16)	M_{max}/Pc
0.49	0.0302	-0.033136	- 0.2232	-0.054421	0.2287	0.2075
1.0	-0.034180	-0.024950	-0.049876	0.029918	-0.0049434	0.2118
2.35	-0.014884	0.00075516	0.02570	-0.00098657	0.0001526	0.2019
5.6	9736.5 $\times 10^{-8}$	-207.6 $\times 10^{-8}$	-40798.1 $\times 10^{-8}$	42.62 $\times 10^{-8}$	-13.5 $\times 10^{-8}$	0.2039

ROCK

Closed Forms: $\Phi(0) = 0.27533$ $\Phi''(0) = 0.48237$

$f = \ell/2c$	$\Phi(\ell/2c)$	$\Phi(\ell/c)$	$\Phi'(\ell/2c)$	$\Phi'(\ell/c)$	$\Phi''(\ell/c)$	M_{max}/Pc
2.35	-0.043007	-0.001506	0.02488	0.0046709	-0.0074243	0.2659
3.4	-0.015928	0.00038780	0.019742	-0.00044738	0.00009247	0.2736
5.6	34660.82 $\times 10^{-8}$	-4934.604 $\times 10^{-8}$	-9063.2 $\times 10^{-8}$	1016.18 $\times 10^{-8}$	-1238.566 $\times 10^{-8}$	0.2753

Note: Shear and moment are equal to zero in each case at the ends of the beam. $2.35 = \ell/2c$ is typical for grillage beams on rock in Ketchum's tables.[8]

Free End Effects: The maximum change in per cent of M_{max}/Pc for the finite beam as compared with the infinite beam is less than 4%.

requires beam lengths such as to make $\ell/2c > 1$.[10] An increase in the moment of inertia of the beam, I, has the same effect as a decrease in the Young's modulus of foundation, E, and results in an increased maximum moment in the beam.

The influence of the free ends is noted for $\ell/2c < 3.4$ in the case of rock, and $\ell/2c < 2.35$ in the case of sand foundations. The end effects are practically the same as for an infinitely long beam. Increasing the length of the beam beyond $\ell/2c = 3.4$ for a rock foundation does not increase the maximum moment, contrary to the practical design assumption that the moment varies directly as the beam length. This fact is indicated by the F curves and is often recognized intuitively in practice in designing, for example, bearing plates which must be extended in length for other than structural reasons. In this case an arbitrary equivalent length is assumed acted on by an assumed uniform bearing pressure equal to the load per unit width divided by this arbitrary equivalent length.

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10. "Foundations, Abutments and Footings," by G. A. Hool and W. S. Kinne, McGraw-Hill Book Co., New York, 1923, pp. 245-247.

PARAMETRIC CURVES OF M_{max}/Pc AGAINST $\ell/2c$

BASED ON DATA IN TABLES 1 AND 2

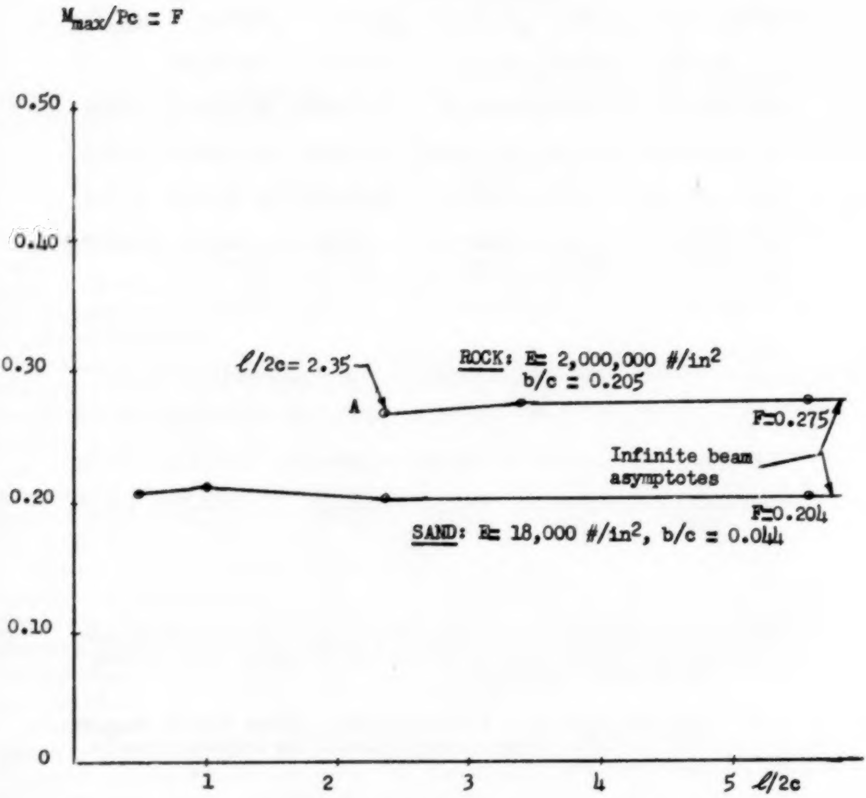

Fig. 5

Note: Point A corresponds to the designs in Ketchum's tables. See Table 1.

The assumption of uniform pressure gives approximately twice the maximum moment determined from the F curves. The design of the beam is therefore conservative when the uniform pressure distribution is assumed. The maximum pressure on the foundation computed by means of the elastic analysis, however, is found to exceed the prescribed allowable bearing pressure considerably, as is demonstrated in the following illustrative problem.

The elastic analysis gives considerably less flexural stress in the beam. Due to the higher bearing pressure under the central portion of the beam, a greater width of contact is desirable in order to avoid local crushing of the foundation.

ILLUSTRATIVE PROBLEM

Consider a 20 I 90, 7' - 5" long, $I = 1550.3 \text{ in}^4$, $b = 3.56$ in., and $S = 155.0 \text{ in}^3$. Take $E_b = 30{,}000 \text{ kips/in}^2$, $E = 2{,}000 \text{ kips/in}^2$, $C_{avg} = 1.06$, $\nu = 1/4$, and $E_b/E = 15$. Compute the fundamental length:

$$c = \left[C(1-\nu^2)\frac{E_b I}{Eb}\right]^{1/3} = \left[1.06(1-\tfrac{1}{4}^2)\,15\,I/b\right]^{1/3} = 2.46(I/b)^{1/3},$$

for all beams on rock. For the particular beam, substitute the moment of inertia, I, and the half flange width b. $\therefore$ $c = 18.6"$.
Hence $b/c = 3.56/18.6 = 0.191$, and $\ell/2c = 7.42 \times 12/2 \times 18.6 = 2.39$. Enter the F curve for rock in Fig. 5 with $\ell/2c = 2.39$ and obtain $F = 0.265$. Note: Most beams in Ketchum's[8] grillage tables average $b/c = 0.205$, and $\ell/2c = 2.35$. Hence there is only a slight reduction in the F value from that of an infinitely long beam.

$$M_{max} = PcF = P(18.6)(0.265) = 4.93\ P.$$

The units are M_{max} in inch-kips, and P in kips. For a steel beam on rock or average concrete, $Eb/E = 15$ and $M_{max} = 0.653P\ \sqrt[3]{I/b}$.

For the assumption of uniform pressure:

$$M_{max} = P\ell/8 = (7.42\times12/8)\ P = 11.1P$$

$$\therefore\ 4.93/11.1 = 44.4\%$$

Note: In this case, which is typical of the grillage beams in Ketchum's tables. The maximum beam moment and stresses are only 44% of those obtained when uniform bearing pressure is assumed.

By equating $M_{max} = 4.93$ P, determined from the F curve, to $M_{max} = P/4\beta_o$ of the spring theory, an equivalent spring constant, k, is determined.

$$4.93\ P = P/4\beta_o \quad , \text{ where } \beta_o = \sqrt[4]{k/4E_b I}$$

$$\beta_o = 1/4\times4.93 = 0.0507\ (1/\text{in})$$

$$\therefore\ k = 4E_b I\,\beta_o^4 = 4\times30{,}000\times1550.3\times(0.0507)^4 \qquad E_b = 30{,}000\ \text{kips/in}^2$$

$$k = 1230\ \text{kips/in}^2 \qquad I = 1550.3\ \text{in}^4$$

Since $\ell = \pi/\beta_o = 62"$ is less than the beam length, the tendency towards uplift is assumed prevented by the foundation sustaining tension. Using the spring theory, shear $V = -E_b \dfrac{Id^3y}{dx^3} = \dfrac{-Pe^{-\beta_o x}\cos\beta_o x}{2}$,

$$p = \frac{dV}{dx} = \frac{P\beta_o}{2} \cdot e^{-\beta_o x}\,(\sin\beta_o x + \cos\beta_o x) = ky\ .$$

The function $\varphi = e^{-\beta_o x}(\sin\beta_o x + \cos\beta_o x)$ has been tabulated by Timoshenko[11] and Hetenyi.[12]

11. "Strength of Materials," by S. Timoshenko, D. Van Nostrand Company, New York, 1941, Part II, p. 5.
12. "Beams on Elastic Foundation," by M. Hetenyi, U. of Michigan Press, Ann Arbor, Michigan, 1946, p. 217.

Now, under the load, $x = 0$ and $p = p_{max} = P\beta_o/2 = P(0.0507)/2$
$= 0.0254$ P kips per inch of beam length.

$\therefore$ max unit pressure $= p_{max}/2b = 0.0254P/7.13 = 0.00356P$ (kips/in^2)

For a maximum allowable pressure of 0.500 kips/in^2, the maximum load $P = 140$ kips.

For a maximum allowable bending stress of 18 kips/in^2, $M = fS$ or $4.93P = 18 \times 155.0$, $P = 566$ kips

Using this elastic analysis, the allowable bearing pressure governs the maximum load to be imposed on the beam. Under the uniform pressure assumption

$$P_{allow.brg.} = \overset{f}{0.500} \times \overset{2b}{7.13} \times (7.42 \times 12) = 317 \text{ kips}$$

$$P_{allow.bend.} = \frac{18 \times 155 \times 8}{7.42 \times 12} = 251 \text{ kips}$$

Here the maximum bending stress in the beam governs the maximum load to be imposed.

Now at $\beta_o x = 1$, however, the maximum pressure drops to 50.8% of its value under the load.[11] $x = 1/0.0507 = 19.7''$. Lower values occur beyond a distance equal to about one fourth the beam length from the load.

In this example it is shown that if the beam is selected based on a specified permissible uniform bearing pressure, the flexural stress computed by the elastic analysis is low (about 1/2), and the actual maximum bearing pressure is higher (about twice) than the original permissible bearing pressure. The deflection under the load using the spring theory is:

$$P\beta_o/2k \; \frac{0.0507P}{2 \times 1230} = 0.0000206P \text{ inches, where P is in kips.}$$

Under the uniform bearing assumption, max. defl. $= \dfrac{(P/2)(\ell/2)^3}{8EI}$

$$\text{max. defl.} = \frac{7.42^3 \times 1728P}{8 \times 16 \times 30{,}000 \times 1550.3} = 0.000119P \text{ inches.}$$

Here less deflection (about 1/6) is indicated when the spring theory is used.

The particular beam selected is adequate to sustain a concentrated load of 140 kips when the elastic analysis is used. The assumption of uniform pressure indicates a permissible load of 250 kips. The results show that from purely elastic considerations the usual treatment of the grillage beam, the primary function of which is to reduce the maximum pressure on the foundation, is considerably in error.

Comparison with Test Results

It is interesting to compare the maximum stress determined by means of the curves of Fig. 5 with some test results stated by de Beer.[13,14] The

13. "Calcul de Poutres reposent sur le sol," by E. E. de Beer, Ann. Trav. publics Belg., Vol. 49, 1948, pp. 393-413, 525-552, 653-691, 721-749, especially p. 740 and p. 746.
14. "Tests for the Determination of the Distribution of Soil Reactions Underneath Beams Resting on Soil," by E. de Beer, Proc. Sec. Intern. Conf. Soil Mechanics and Found. Eng., Vol. 2, 1948, pp. 142-148.

Boussinesq solution was employed by de Beer to compute the stresses in the foundation for the case of an ideal soil treated as homogeneous and isotropic. De Beer used a method of successive approximations to secure a nearly coincident beam and soil displacement pattern. Since de Beer's results are a function of the parameter width/length of the loaded area of the foundation, the results obtained are not general but must be repeated for each ratio.

The parametric curves of the present paper were originally computed to encompass the grillage beam designs shown by Ketchum.[8] The choices of the parameter b/c were originally for the cases of steel beams on rock and for steel beams on ideal sand. The parametric form of these curves, however, allows ready substitution of de Beer's numerical values for the physical quantities concerned. The maximum moment can be quickly estimated.

Use de Beer's values for a stiff T beam: $I_{xx} = 949\ cm^4$; Section Modulus $= 353\ cm^3$; $E_b = 2 \times 10^6\ kg/cm^2$; $\ell = 1$ meter; width $= 0.25$ m.; and $E_{sand} = 544\ kg/cm^2$ (experimental average). Enter the parametric curves of the present paper, Fig. 5, with:

$$c = \left[C(1-\nu^2)\ E_b\ I/\ Eb\right]^{1/3} = \left[1.06\ (1-\tfrac{1}{4}^2)\ \frac{2\times10^6\times949}{544\times12.5}\right]^{1/3}$$

$$c = 60.8\ cm; \qquad b/c = 12.5/60.8 = 0.206; \quad \ell/2c = \frac{100}{2\times60.8} = 0.825$$

It is seen by linear extrapolation of the rock curve that $F = M_{max}/Pc \cong 0.25$ and $M_{max} = PcF = 60.8 \times 0.25P = 15.2P$ cm-kg. For $P = 2500$ kg, which is an average load value in de Beer's tables, $\sigma = 15.2 \times 2500/353 = 108\ kg/cm^2$.

			stated by de Beer		
Stiff Tee Beam	Unif. Pressure Assumption	This Paper	K Method	2nd Degree Parabola	Strain Gages
stress kg/cm^2	88.5	108	84.5	101	112.2 115

In the case of the flexible T beam considered by de Beer the moment of inertia is reduced to $54.7\ cm^4$, section modulus $= 48.7\ cm^3$ and the other quantities remain the same. The results for $P = 2500$ kg are as follows:

			stated by de Beer		
Flexible Tee Beam	Unif. Pressure Assumption	This Paper	K Method	2nd Degree Parabola	Strain Gages
stress kg/cm^2	642	460	444	414	381

CONCLUSIONS

The parametric curves of M_{max}/Pc against $\ell/2c$ shown are for two values of b/c which correspond to the numerical values of $E_{rock} = 2{,}000{,}000\ \#/in^2$ and $E_{sand} = 18{,}000\ \#/in^2$. From these curves determinations are made of the maximum beam moments which do not contain the spring constant. For any significant change in one or more of the elastic or geometric quantities embodied in the ratios M/Pc, $\ell/2c$ and b/c, the parametric curves can be used. It is noted that for the entire range of grillage beam designs shown in Ketchum's tables,[8] the value of b/c is substantially constant for any

particular value of E. Using $E = 2,000,000$ #/in^2, for example, $b/c = 0.205$.

A non-experimental determination of k is readily obtained by equating the maximum moment in the spring foundation. Using this value of k, basic problems, such as grillage beams, can be treated and their solution compared with results obtained by other means. Instead of using expressions for deflection, slope, moment and shear containing an experimental constant k, the formulas of this paper employ the elastic constants E and ν which are experimentally determined. The known values of E and ν characterize only the foundation material and are independent of the structure to be imposed.

By use of the expression for the moment in the infinite beam under a single concentrated load the maximum moment for a beam of arbitrary length subjected to any loading can be developed by superposition.

The computations of this paper indicate that the greater pressures present in the vicinity of the concentration result in considerably less bending stress in the beam. The larger maximum foundation pressure is not substantially reduced by the prevalent practice of increasing the beam length.

APPENDIX. NOTATION

α (dimensionless) $= \lambda c$

b (in.) = 1/2 flange width of beam

b/c (dimensionless) = parameter of problem: ratio of 1/2 flange width of beam, 2b, to a fundamental length, c. This parameter is substantially constant for typical grillage tables for each type of foundation (i.e., for any particular E).

β (dimensionless) $= b\alpha/c$ parameter.

β_o (1/in.) $= \sqrt[4]{k/4E_oI}$ reciprocal of the fundamental length arising in the spring theory. Sometimes termed 'characteristic' or 'damping factor.'

c (in.) = fundamental length $\left[C\ (1 - \nu^2) \frac{E_bI}{Eb} \right]^{1/3}$

C (dimensionless) = coefficient representing effect of changing distribution of loading across width 2b. $C = 1.13$ for uniform deflection across width 2b. $C = 1.06$ used as average in the computations. $C = 1$ for uniform pressure across width 2b.

E_b (#/in^2) = Young's modulus for the beam.

E (#/in^2) = Young's modulus for the foundation.

$F = M_{max}/Pc$ = dimensionless quantity by which Pc is multiplied to obtain, M_{max}, the moment under the concentration P. F is determined after entering the curves in Fig. 5 with values of $\ell/2c$ and b/c.

f (#/in^2) = allowable stress.

$\gamma = x/c$ = parameter

I (in^4) = moment of inertia of the beam.

k (#/in. length/ in. deflection) = spring modulus.

k_o (#/in^3: #/in. length/in. deflection/unit width of beam) = spring modulus of the foundation based on unit width of beam: sometimes termed 'coefficient of subgrade reaction.'

ℓ (in.) = length of finite beam.

L_{max} (#/in. of beam length) $= 2q_ob$ = maximum load on the foundation surface.

$L_{avg.}$ (#/in. of beam length) = average load on the foundation surface.

ℓ/c, $\ell/2c$ = parameter of problem.

λ (1/in.) = reciprocal of fundamental wave length of loading in beam length direction.

M (in-#) = moment in beam due to load P.

M_o(in-#) = unknown annulling moments applied at those points of the infinite beam, corresponding to the end points of the finite beam, so as to give the free end conditions.

ν (dimensionless) = Poisson's ratio of the foundation. This is defined as the ratio of lateral contraction to longitudinal extension for a particular material. Assumed constant = 1/4 in computations.

P (#) = concentrated load at center of beam.

p (#/in. of beam length) = reaction of beam on foundation.

p(x) (#/in. of beam length) = load distribution along the beam.

Q_o (#) = unknown annulling shear forces applied at those points of the infinite beam, corresponding to the end points of the finite beam, so as to give the free end conditions.

q_o (#/in^2) = prescribed constant pressure distribution across the flange width.

S (in^3) = section modulus of beam.

V (#) = dM/dx = shear.

$W_o(y)$ (in.) = maximum deflection of foundation surface along a section across the flange width of beam.

$W_{avg.}$ (in.) = average deflection of the foundation surface across the flange.

$$\Phi(\xi) = \frac{1}{\pi}\int_0^\infty \frac{\alpha \cos \alpha\xi \, d\alpha}{\alpha^3 + \psi(\beta)}$$

= dimensionless quantity by which Pc is multiplied to obtain the moment at any point in an infinite beam due to a concentrated load, P, at the origin.

$$\Phi'(\xi) = \frac{d\Phi(\xi)}{d\xi} = c\,\frac{d\Phi}{dx} = -\frac{1}{\pi}\int_0^\infty \frac{\alpha^2 \sin \alpha\xi \, d\alpha}{\alpha^3 + \psi(\beta)}$$

= dimensionless quantity by which P is multiplied to obtain the shear at any point in an infinite beam due to a concentrated load, P, at the origin; dimensionless quantity by which the moment M_o is multiplied to obtain the moment at any point in an infinite beam due to M_o at the origin.

$$\Phi''(\xi) = \frac{d^2\Phi(\xi)}{d\xi^2} = c^2\,\frac{d^2\Phi}{dx^2} = -\frac{1}{\pi}\int_0^\infty \frac{\alpha^3 \cos \alpha\xi \, d\alpha}{\alpha^3 + \psi(\beta)}$$

= dimensionless quantity by which M_o/c is multiplied to obtain the shear at any point in an infinite beam due to M_o at the origin.

x (in.) = distance of any point along the beam from the origin.

y (in.) = vertical deflection of any point along the beam.

$\psi(\beta)$ (dimensionless) = a hyperbolic function obtained by fitting by means of least squares to Biot's asymptotic expression and tabular values. It relates L_{max} to $W_{avg.}$

PROCEEDINGS PAPERS

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AUGUST: 466(HY), 467(HY), 468(ST), 469(ST), 470(ST), 471(SA), 472(SA), 473(SA), 474(SA), 475(SM), 476(SM), 477(SM), 478(SM)c, 479(HY)c, 480(ST)c, 481(SA)c, 482(HY), 483(HY).

SEPTEMBER: 484(ST), 485(ST), 486(ST), 487(CP)c, 488(ST)c, 489(HY), 490(HY), 491(HY)c, 492(SA), 493(SA), 494(SA), 495(SA), 496(SA), 497(SA), 498(SA), 499(HW), 500(HW), 501(HW)c, 502(WW), 503(WW), 504(WW)c, 505(CO), 506(CO)c, 507(CP), 508(CP), 509(CP), 510(CP), 511(CP).

OCTOBER: 512(SM), 513(SM), 514(SM), 515(SM), 516(SM), 517(PO), 518(SM)c, 519(IR), 520(IR), 521(IR), 522(IR)c, 523(AT)c, 524(SU), 525(SU)c, 526(EM), 527(EM), 528(EM), 529(EM), 530(EM)c, 531(EM), 532(EM)c, 533(PO).

NOVEMBER: 534(HY), 535(HY), 536(HY), 537(HY), 538(HY)c, 539(ST), 540(ST), 541(ST), 542(ST), 543(ST), 544(ST), 545(SA), 546(SA), 547(SA), 548(SM), 549(SM), 550(SM), 551(SM), 552(SA), 553(SM)c, 554(SA), 555(SA), 556(SA), 557(SA).

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FEBRUARY: 608(WW), 609(WW), 610(WW), 611(WW), 612(WW), 613(WW), 614(WW), 615(WW), 616(WW), 617(IR), 618(IR), 619(IR), 620(IR), 621(IR)c, 622(IR), 623(IR), 624(HY)c, 625(HY), 626(HY), 627(HY), 628(HY), 629(HY), 630(HY), 631(HY), 632(CO), 633(CO).

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c. Discussion of several papers, grouped by Divisions.

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